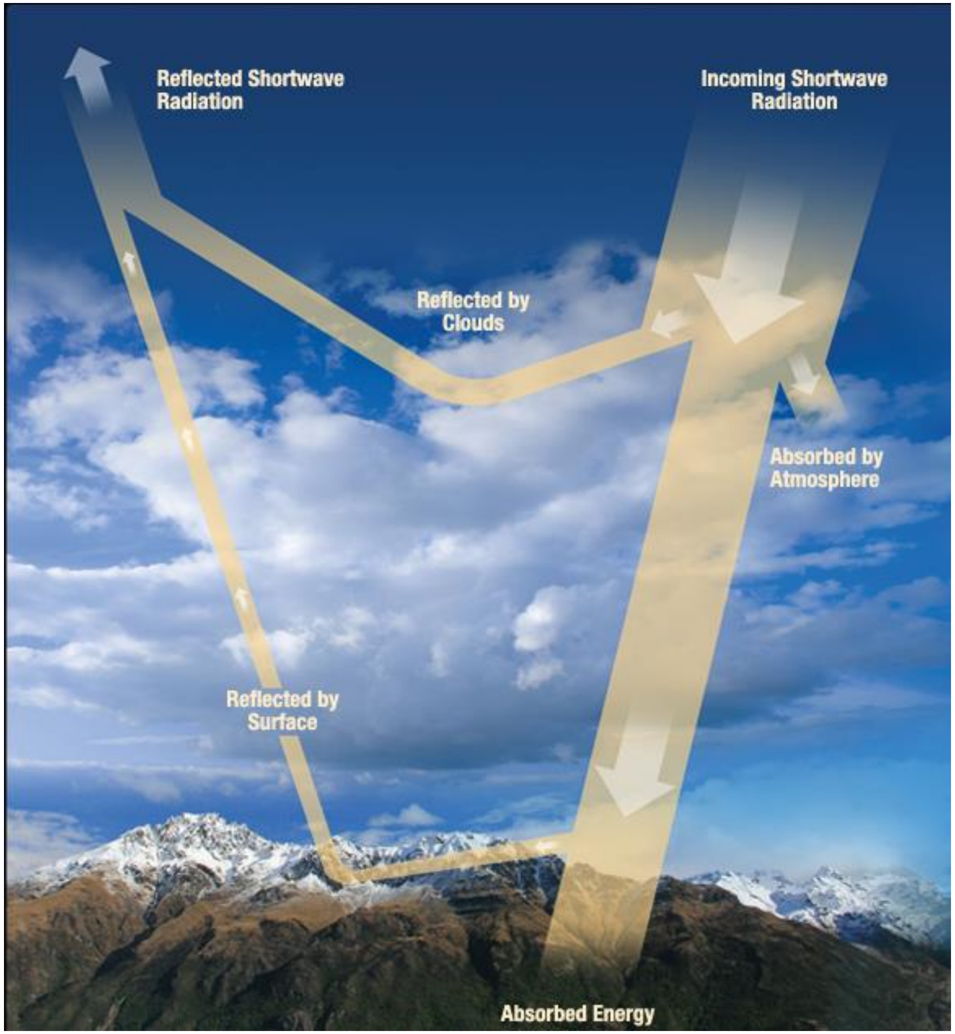
Lesson 10:
EQUATION OF THE RADIATIVE TRANSFER



Course: Laboratory of Atmospheric Remote Sensing
Laurea Magistrale in Atmospheric Science and Technology

Content

- Equation of Radiative Transfer: definition, extinction, emission and scattering, general radiative transfer equation without any coordinate system imposed
- Beer–Lambert law
- Transmissivity and absorptivity, optical thickness
- Schwarzschild's equation
- Plane-parallel atmosphere: definition, applications, radiative transfer equation for plane-parallel atmosphere
- Upward radiation at the TOA and at the surface

Reading material:

- Liou K. N., An introduction to atmospheric radiation. Chps. 1-2.
- Wallace J.M and Hobbs P.V., Atmospheric Science: An Introduction survey. Chpt. 4.

Equation of Radiative Transfer (1)

It describes the variations that a band of electromagnetic radiation that propagates through the atmosphere.
The scattering and absorption of radiation by gas molecules and aerosols contribute to the extinction of the solar and terrestrial radiation depending on:

- the intensity of the radiation at that point along the ray path
- the local concentration of the gases and/or particles that are responsible for the absorption and scattering
- the effectiveness of the absorbers or scatterers

If the radiation is monochromatic and the atmosphere is absorbing and diffusing at that wavelength, the bundle will undergo a variation in dI_λ crossing an infinitesimal layer proportional to the incoming radiance and the amount of matter crossed.

Interaction radiation – layer of atmosphere

<u>Reduction</u> (**extinction**) $\Rightarrow\ dI_\lambda < 0$
<u>Increasing</u> (**emission + scattering**) $\Rightarrow\ dI_\lambda > 0$

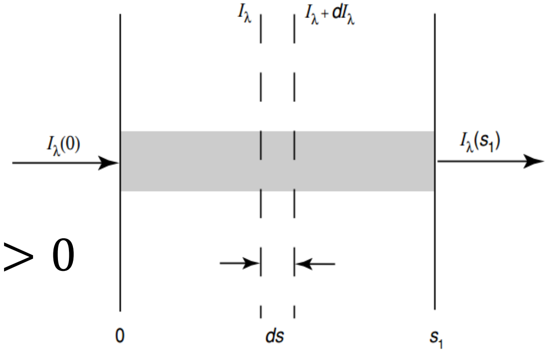


Figure 1.13 Depletion of the radiant intensity in traversing an extinction medium.

Equation of Radiative Transfer (2)

ρ = density of the material
k_λ = mass extinction cross section (in units of area per mass) for radiation of wavelength λ
M = mass of particles
A = areal cross section of each particle

If the radiation is monochromatic and the atmosphere is absorbing and diffusing at that wavelength, the bundle will undergo a variation in dI_λ crossing an infinitesimal layer proportional to the incoming radiance and the amount of matter crossed.
The radiance of the bundle will also have an increase due to the emission of the infinitesimal layer, which will be proportional to the amount of matter.

$dI_\lambda \propto dM$

$$dI_\lambda = -\tilde{k}_\lambda I_\lambda dM + \tilde{j}_\lambda dM$$

Equation of Radiative Transfer (3)

Suppose that the layer crossed by the radiation strip is thick dS and area A

$$dM = \rho dV = \rho A dS$$

$$dI_\lambda = -\tilde{k}_\lambda \rho A I_\lambda dS = -k_\lambda \rho I_\lambda ds$$

We can define $k_\lambda = \tilde{k}_\lambda A$ and $j_\lambda = \tilde{j}_\lambda A$

$$[k_\lambda \rho ds] = [1] \qquad \Rightarrow [k_\lambda] = \left[\frac{1}{\rho ds}\right] = \frac{L^3}{M}\frac{1}{L} = \frac{L^2}{M}$$

$$dI_\lambda = -k_\lambda \rho I_\lambda dS + j_\lambda \rho dS = (-k_\lambda I_\lambda + j_\lambda)\rho dS = (-I_\lambda + \frac{j_\lambda}{k_\lambda})k_\lambda \rho dS$$

We can define the **source function** as

$$J_k = \frac{j_\lambda}{k_\lambda}$$

Equation of Radiative Transfer (4)

$$dI_\lambda = (-I_\lambda + J_\lambda)k_\lambda \rho dS$$

$$\frac{dI_\lambda}{k_\lambda \rho ds} = -I_\lambda + J_\lambda$$

General radiative transfer equation without any coordinate system imposed

In atmospheric conditions, both emission and scattering contribute to the source function.

Equation of Radiative Transfer (5)

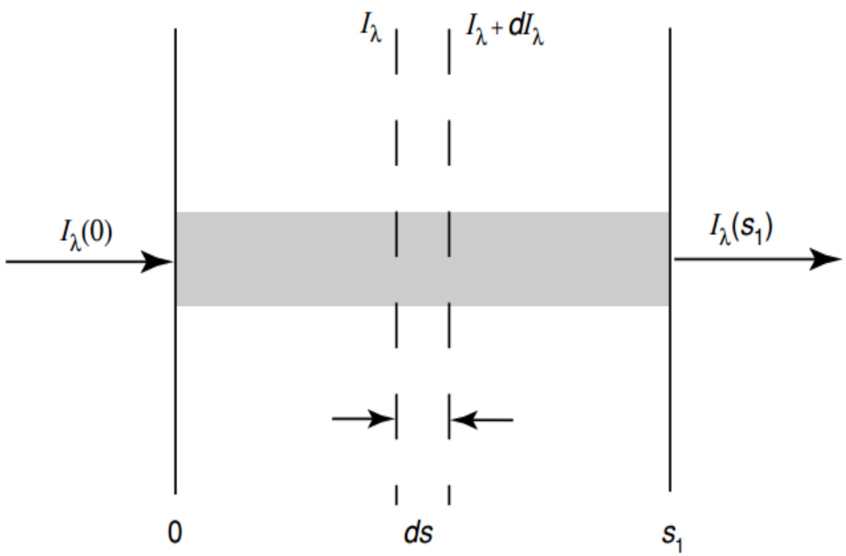


Figure 1.13 Depletion of the radiant intensity in traversing an extinction medium.

The general equation of radiative transfer describes the variation that the radiance undergoes crossing a layer of the atmosphere of thickness ds.

To know the radiance that comes out of a layer of finite thickness s_1 it is necessary to integrate the equation from point 0 to point s_1 knowing the variance at the entrance to the layer.

Beer–Lambert Law (1)

To study the problem we start from the particular case in which the source function is null.
This is the case in which we are looking directly at the Sun at short wavelengths, so the contribution of atmospheric emission is zero and that of diffusion negligible. The equation becomes

Considering solar radiation (wavelengths from about 0.2 to 5 μm):

- Emission contributions from the earth–atmosphere system can be generally neglected
- Diffuse radiation produced by multiple scattering can be neglected

$$J_\lambda = 0 \;\Rightarrow\; \frac{dI_\lambda}{k_\lambda \rho ds} = -I_\lambda$$

$$\frac{dI_\lambda}{I_\lambda} = -k_\lambda \rho ds$$

Beer–Lambert Law (2)

Let the incident intensity at $s = 0$ be $I_\lambda(0)$ and integrate between 0 and a generic point s

$$\int_{I_\lambda(0)}^{I_\lambda(s)} d\, ln I_\lambda = -\int_0^s k_\lambda(s')\rho(s')ds'$$

$$ln\frac{I_\lambda(s)}{I_\lambda(0)} = -\int_0^s k_\lambda(s')\rho(s')ds'$$

$$\boxed{I_\lambda(s) = I_\lambda(0)\exp\left(-\int_0^s k_\lambda(s')\rho(s')ds'\right)}$$

Beer–Lambert Law

Beer–Lambert Law (3)

Considering:

- homogeneous medium
- $k_\lambda\ (s) = k_\lambda$ i.e. independent of the distance s

And writing

$$\mathrm{u}(s) = \int_0^{s_1} \rho(s')ds'$$

where u(s) is the path length

The Beer-Lambert law can be rewriteen as

$$\boxed{I_\lambda(s) = I_\lambda(0)e^{-k_\lambda u(s)}}$$

Beer-Lambert Law

The decrease in the radiant intensity traversing a homogeneous extinction medium is in accord with the simple exponential function whose argument is the product of the mass extinction cross section and the path length.

Transmissivity and absorptivity

$$T_\lambda = \frac{I_\lambda(s)}{I_\lambda(0)} = e^{-k_\lambda u(s)}$$

Monochromatic Transmissivity

$$A_\lambda = 1 - T_\lambda = 1 - e^{-k_\lambda u(s)}$$

Monochromatic Absorptivity (for nonscattering medium)

We can define the Monochromatic Reflectivity R_λ, which is the ratio of the reflected (backscattered) intensity to the incident intensity.

On the basis of the conservation of energy, we must have:

$$T_\lambda + A_\lambda + R_\lambda = 1$$

for the transfer of radiation through a scattering and absorbing medium.

Optical thickness

The radiative transfer equation, and its approximation with the Beer-Lambert law, is generally expressed as a function of a new variable, the optical thickness:

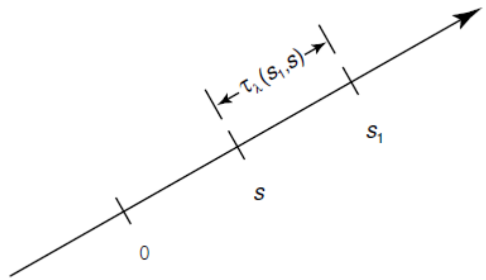


Figure 1.14 Configuration of the optical thickness τ_λ defined in Eq. (1.4.15).

$$\tau_\lambda(s_1, s) = \int_s^{s_1} k_\lambda \rho ds'$$

τ_λ = monochromatic optical thickness of the medium between points s and s_1

$$\tau_\lambda(s_1, s_1) = 0$$

$$\tau_\lambda(0, s_1) = \tau_\lambda(s_1) = \int_0^{s_1} k_\lambda \rho ds'$$

$$\tau_\lambda(s) = \int_s^{s_1} k_\lambda \rho ds' = -\int_{s_1}^{s} k_\lambda \rho ds'$$

$$d\tau_\lambda(s_1, s) = -k_\lambda \rho ds$$

The optical thickness collects the effect of the extinction of the radiation of the layer that goes from the generic point s to the extreme s_1 of the layer along the direction of propagation.
τ_λ decreases as s increases and its infinitesimal variation is opposite to ds.

Schwarzschild's Equation (1)

We get Beer's Lambert law using optical thickness instead of s

$$dI_\lambda = -I_\lambda k_\lambda \rho ds = I_\lambda d\tau_\lambda \qquad \frac{dI_\lambda}{I_\lambda} = d\tau_\lambda$$

$$\int_{I_\lambda(0)}^{I_\lambda(s)} dln I_\lambda = \int_{\tau_\lambda(0)}^{\tau_\lambda(s)} d\tau_\lambda = -(\tau_\lambda(0) - \tau_\lambda(s))$$

$$I_\lambda(s) = I_\lambda(0) exp\{-(\tau_\lambda(0) - \tau_\lambda(s))\}$$

The argument of the exponential is the difference between the total optical thickness of the layer and the optical thickness between the point s and the end of the layer, i.e. the optical thickness from 0 to s.
The optical thickness is used instead of the spatial coordinate in the law of radiative transfer

$$\frac{dI_\lambda}{k_\lambda \rho ds} = -I_\lambda + J_\lambda \quad \Rightarrow \quad -\frac{dI_\lambda}{d\tau_\lambda} = -I_\lambda(\tau_\lambda) + J_\lambda(\tau_\lambda)$$

The radiance and the source term are computed in τ_λ instead of s

Schwarzschild's Equation (2)

Now let's study the case in which the source function contributes only the emission and not the scattering with monochromatic radiation and in local thermodynamic equilibrium.

Ipothesis: only molecules (no scattering)

k_λ= absorption coefficient
T_B= Brilliance temperature that is the temperature at which the atmosphere must be in the point τ to emit as a black body.

The source function is given by the Planck function and can be expressed by:

$$J_\lambda = B_\lambda(T)$$

$$-\frac{dI_\lambda}{d\tau_\lambda} = -I_\lambda(\tau_\lambda) + B_\lambda(T_B(\tau_\lambda)) \quad \Rightarrow \quad -\frac{dI}{d\tau} = -I(\tau) + B[T_B(\tau)]$$

To simplify the notation we drop the subscript λ.

The **first term** in the right-hand denotes the reduction of the radiant intensity due to absorption, whereas the **second term** represents the increase in the radiant intensity arising from blackbody emission of the material.

Schwarzschild's Solution (3)

$$-\frac{dI}{d\tau} = -I(\tau) + B[T_B(\tau)]$$

Multiplying by a factor $e^{-\tau}d\tau$:

$$-dIe^{-\tau} = -I(\tau)e^{-\tau}d\tau + B[T_B(\tau)]e^{-\tau}d\tau$$

$$-dIe^{-\tau} + I(\tau)e^{-\tau}d\tau = B[T_B(\tau)]e^{-\tau}d\tau$$

The two terms to the left of the equal can be collected in a single differential since

$$d(Ie^{-\tau}) = dIe^{-\tau} - I(\tau)e^{-\tau}d\tau$$

$$-d(Ie^{-\tau}) = B[T_B(\tau)]e^{-\tau}d\tau$$

Schwarzschild's Solution (4)

Integrating the thickness ds from 0 to s_1.

The integration limits of the integral on the left of the equal are

$$I(0)e^{-\tau(0,s_1)} = I(0)e^{-\tau(s_1)} \quad \text{and} \quad I(s_1)e^{-\tau(s_1,s_1)} = I(s_1)$$

The integration limits of the integral on the right of the equal are

$$\tau(0,s_1) = \tau(s_1) \quad \text{and} \quad \tau(s_1,s_1) = 0$$

$$-\int_{I(0)e^{-\tau(s_1)}}^{I(s_1)} d(Ie^{-\tau}) = \int_{\tau(s_1)}^{0} B_\lambda[T_B(\tau)]e^{-\tau}d\tau$$

$$I(s_1) = I(0)e^{-\tau(s_1)} + \int_0^{\tau(s_1)} B_\lambda[T_B(\tau)]e^{-\tau}d\tau$$

Schwarzschild's Equation

Schwarzschild's Solution (5)

$$I(s_1) = I(0)e^{-\tau(s_1)} + \int_0^{\tau(s_1)} B_\lambda[T_B(\tau)]e^{-\tau} d\tau$$

- The **first term** represents the absorption attenuation of the radiant intensity by the medium.
- The **second term** denotes the emission contribution from the medium along the path from 0 to s_1.

If the temperature and density of the medium and the associated absorption coefficient along the path of the beam are known, the Schwarzschild's Equation can be integrated numerically to yield the intensity at the point s_1.

Fundamental equations

1. Solar spectrum without multiple scattering:
   Beer-Lambert Equation

$$I_\lambda(s) = I_\lambda(0)\exp\left(-\int_0^s k_\lambda \rho ds\right)$$

2. Terrestrial spectrum – only molecules:
   Schwarzschild's Equation

$$I(s_1) = I(0)e^{-\tau(s_1)} + \int_0^{\tau(s_1)} B_\lambda[T_B(\tau)]e^{-\tau}d\tau$$

Plane-Parallel Atmosphere (1)

For many atmospheric radiative transfer applications, it is physically appropriate to consider that the atmosphere in localized portions is plane-parallel such that variations in the intensity and atmospheric parameters (temperature and gaseous profiles) are permitted only in the vertical direction (i.e., height and pressure).

It is frequently used to avoid the computational complications related to the sphericity of the Earth and the horizontal variations of the parameters. It is suitable for scenarios where the horizontal dimensions are smaller than the vertical ones.

In this case, it is convenient to measure linear distances normal to the plane of stratification.

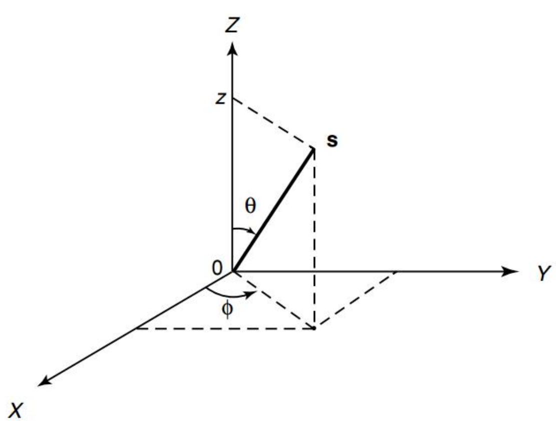


Figure 1.15 Geometry for plane-parallel atmospheres where θ and ϕ denote the zenith and azimuthal angles, respectively, and **s** represents the position vector.

It involves that each variable depends only on z.
It means that the atmosphere is horizontally homogeneous and isotropic.
In this case, the path of the radiation can be projected onto the vertical axis using the zenith angle.

Plane-Parallel Atmosphere (2)

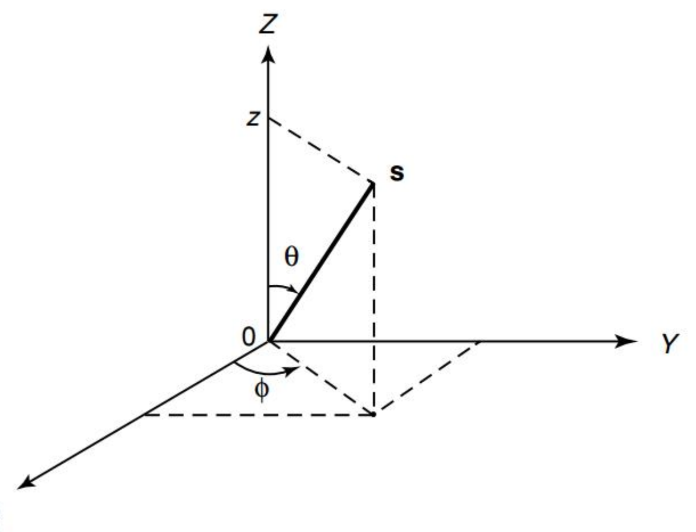


Figure 1.15 Geometry for plane-parallel atmospheres where θ and ϕ denote the zenith and azimuthal angles, respectively, and **s** represents the position vector.

θ = inclination to the upward normal
ϕ = azimuthal angle in reference to the x axis
$\mu \equiv cos\theta$

Upward radiation: $0 \leq \phi \leq 2\pi$ and $0 \leq \theta \leq \frac{\pi}{2}$ and $\mu \geq 0$
Downward radiation: $0 \leq \phi \leq 2\pi$ and $\frac{\pi}{2} \leq \theta \leq \pi$ and $\mu \leq 0$
$f(\mathrm{x,y,z}) = f(\mathrm{z})$
$k_\lambda(\mathrm{x,y,z}) = k_\lambda(\mathrm{z})$

$dz = cos\theta ds = \mu dz' = \mu ds$

Plane-Parallel Atmosphere (2)

$$\cos\theta \frac{dI(z;\theta,\phi)}{k\rho ds} = -I(z;\theta,\phi) + J(z;\theta,\phi)$$

Introducing the **normal optical thickness $\boldsymbol{\tau}$**, measured downward from the outer boundary (TOA):

$$\tau = \int_z^\infty k(z')\rho(z')dz'$$

We can rewrite the equation of the radiative transfer as

$$\mu \frac{dI(\tau;\mu,\phi)}{d\tau} = I(\tau;\mu,\phi) - J(\tau;\mu,\phi)$$

Radiative transfer equation for plane-parallel atmosphere

Solution for the Plane-Parallel Equation (1)

This equation can be formally solved by separating it into two components:

- one for the radiances directed upwards in which $\mu > 0$
- one for the radiances downwards in which $\mu < 0$

The optical thickness value will vary between that at the TOA, which is zero, and that at the surface (τ^*)

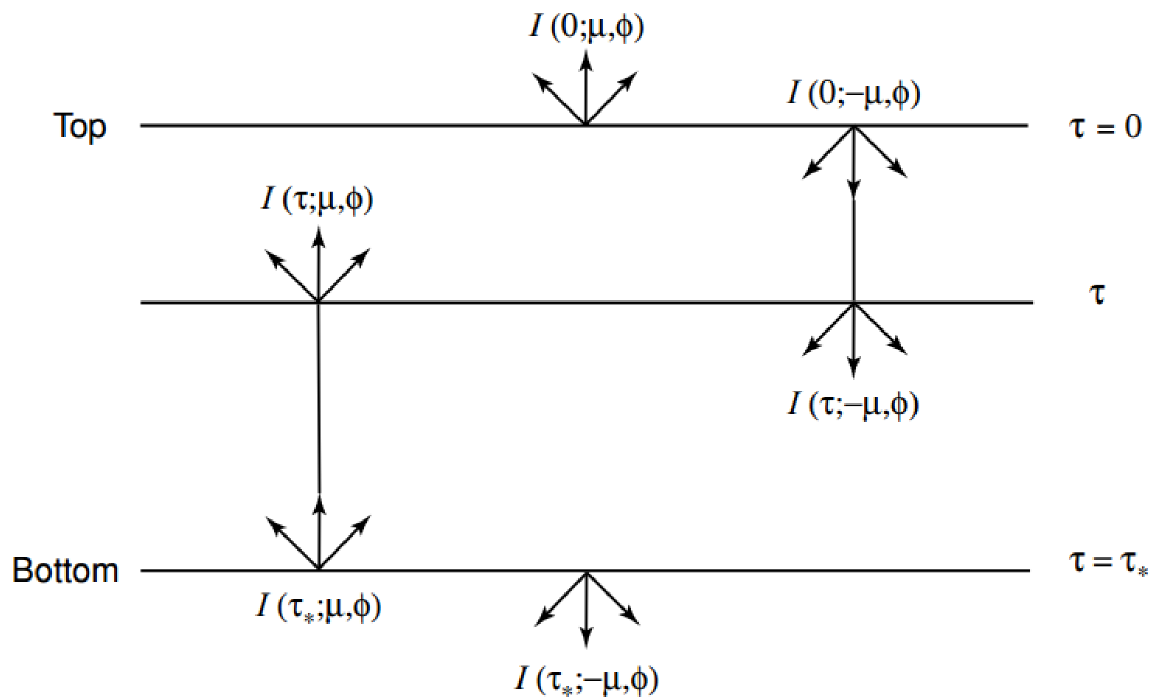


Figure 1.16 Upward (μ) and downward ($-\mu$) intensities at a given level τ and at top ($\tau = 0$) and bottom ($\tau = \tau_*$) levels in a finite, plane-parallel atmosphere.

Solution for the Plane-Parallel Equation (2)

θ variation limited between 0 and $\pi/_2$ (so $\mu > 0$)

In the equation for the downward radiation change μ in – μ.

$$(\uparrow) \quad -\mu\frac{dI(\tau;\mu,\phi)}{d\tau} = -I(\tau;\mu,\phi) + J(\tau;\mu,\phi)$$

$$(\downarrow) \quad \mu\frac{dI(\tau;-\mu,\phi)}{d\tau} = I(\tau;-\mu,\phi) - J(\tau;-\mu,\phi)$$

Can be solved to give the upward and downward intensities for a finite atmosphere that is bounded on two sides at $\tau = 0$ and $\tau = \tau^*$ as illustrated in the figure given in the previous slide.

Solution for the Plane-Parallel Equation (3)

Let's multiply the equation (↑) by a factor $e^{-\tau/\mu} d(\tau/\mu)$
and the equation (↓) by a factor $e^{\tau/\mu} d(\tau/\mu)$

$$\begin{cases} (\uparrow) \quad -\mu \dfrac{dI}{d\tau} e^{-\tau/\mu} d(\tau/\mu) = -I(\tau,\mu,\phi) e^{-\tau/\mu} d(\tau/\mu) + J(\tau,\mu,\phi) e^{-\tau/\mu} d(\tau/\mu) \\ (\downarrow) \quad \mu \dfrac{dI}{d\tau} e^{\tau/\mu} d(\tau/\mu) = -I(\tau,-\mu,\phi) e^{\tau/\mu} d(\tau/\mu) + J(\tau,-\mu,\phi) e^{\tau/\mu} d(\tau/\mu) \end{cases}$$

$$\begin{cases} (\uparrow) \quad -dI e^{-\tau/\mu} = -I(\tau,\mu,\phi) e^{-\tau/\mu} d(\tau/\mu) + J(\tau,\mu,\phi) e^{-\tau/\mu} d(\tau/\mu) \\ (\downarrow) \quad dI e^{\tau/\mu} = -I(\tau,-\mu,\phi) e^{\tau/\mu} d(\tau/\mu) + J(\tau,-\mu,\phi) e^{\tau/\mu} d(\tau/\mu) \end{cases}$$

$$\begin{cases} (\uparrow) \quad -dI e^{-\tau/\mu} + I(\tau,\mu,\phi) e^{-\tau/\mu} d(\tau/\mu) = J(\tau,\mu,\phi) e^{-\tau/\mu} d(\tau/\mu) \\ (\downarrow) \quad dI e^{\tau/\mu} + I(\tau,-\mu,\phi) e^{\tau/\mu} d(\tau/\mu) = J(\tau,-\mu,\phi) e^{\tau/\mu} d(\tau/\mu) \end{cases}$$

$$\begin{cases} (\uparrow) \quad -d\left(I e^{-\tau/\mu}\right) = J(\tau,\mu,\phi) e^{-\tau/\mu} d(\tau/\mu) \\ (\downarrow) \quad d\left(I e^{\tau/\mu}\right) = J(\tau,-\mu,\phi) e^{\tau/\mu} d(\tau/\mu) \end{cases}$$

Solution for the Plane-Parallel Equation (4)

Let's integrate the equation (↑) from the surface (τ*) up to a generic level τ and the equation (↓) from the TOA (τ=0) down to a generic level τ

$$\begin{cases} (\uparrow) \quad -\displaystyle\int_{I(\tau^*)e^{-\tau^*/\mu}}^{I(\tau)e^{-\tau/\mu}} d\left(Ie^{-\tau'/\mu}\right) = \int_{\tau^*}^{\tau} J(\tau',\mu,\phi)e^{-\tau'/\mu} d\left(\tau'/\mu\right) \\ (\downarrow) \quad \displaystyle\int_{I(0)}^{I(\tau)e^{\tau/\mu}} d\left(Ie^{\tau'/\mu}\right) = \int_{0}^{\tau} J(\tau',-\mu,\phi)e^{\tau'/\mu} d\left(\tau'/\mu\right) \end{cases}$$

$$\begin{cases} (\uparrow) \quad I(\tau^*)e^{-\tau^*/\mu} - I(\tau)e^{-\tau/\mu} = \displaystyle\int_{\tau^*}^{\tau} J(\tau',\mu,\phi)e^{-\tau'/\mu} d\left(\tau'/\mu\right) \\ (\downarrow) \quad I(\tau)e^{\tau/\mu} - I(0) = \displaystyle\int_{0}^{\tau} J(\tau',-\mu,\phi)e^{\tau'/\mu} d\left(\tau'/\mu\right) \end{cases}$$

Solution for the Plane-Parallel Equation (5)

Let's multiply the equation (↑) by a factor $e^{\tau/\mu}$
and the equation (↓) by a factor $e^{-\tau/\mu}$

$$\begin{cases} (\uparrow) \quad I(\tau^*)e^{-\tau^*/\mu}e^{\tau/\mu} - I(\tau)e^{-\tau/\mu}e^{\tau/\mu} = \int_{\tau^*}^{\tau} J(\tau',\mu,\phi)e^{-\tau'/\mu}e^{\tau/\mu}d\left(\tau'/\mu\right) \\ (\downarrow) \quad I(\tau)e^{\tau/\mu}e^{-\tau/\mu} - I(0)e^{-\tau/\mu} = \int_{0}^{\tau} J(\tau',-\mu,\phi)e^{\tau'/\mu}e^{-\tau/\mu}d\left(\tau'/\mu\right) \end{cases}$$

$$\begin{cases} (\uparrow) \quad I(\tau^*)e^{-\frac{\tau^*-\tau}{\mu}} - I(\tau) = \int_{\tau^*}^{\tau} J(\tau',\mu,\phi)e^{-\frac{\tau'-\tau}{\mu}}d\left(\tau'/\mu\right) \\ (\downarrow) \quad I(\tau) - I(0)e^{-\tau/\mu} = \int_{0}^{\tau} J(\tau',-\mu,\phi)e^{-\frac{\tau-\tau'}{\mu}}d\left(\tau'/\mu\right) \end{cases}$$

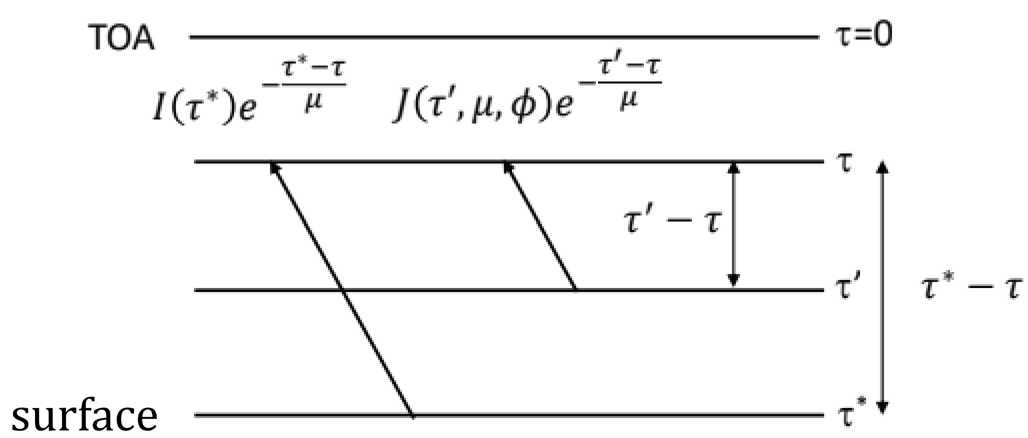
$$\begin{cases} (\uparrow) \quad I(\tau) = I(\tau^*)e^{-\frac{\tau^*-\tau}{\mu}} + \int_{\tau}^{\tau^*} J(\tau',\mu,\phi)e^{-\frac{\tau'-\tau}{\mu}}d\left(\tau'/\mu\right) \\ (\downarrow) \quad I(\tau) = I(0)e^{-\tau/\mu} + \int_{0}^{\tau} J(\tau',-\mu,\phi)e^{-\frac{\tau-\tau'}{\mu}}d\left(\tau'/\mu\right) \end{cases}$$

Solution for the Plane-Parallel Equation (6)

$$(\uparrow) \qquad I(\tau) = I(\tau^*)e^{-\frac{\tau^*-\tau}{\mu}} + \int_{\tau}^{\tau^*} J(\tau',\mu,\phi)e^{-\frac{\tau'-\tau}{\mu}} d\left(\tau'/_{\mu}\right)$$

The radiance that arrives at the level τ from below is given by

1. Incoming from the surface, reduced of the transmission between the surface layer and the altitude τ
2. The contributions of the sources in all layers below the level τ, each reduced by its relative transmission up to τ

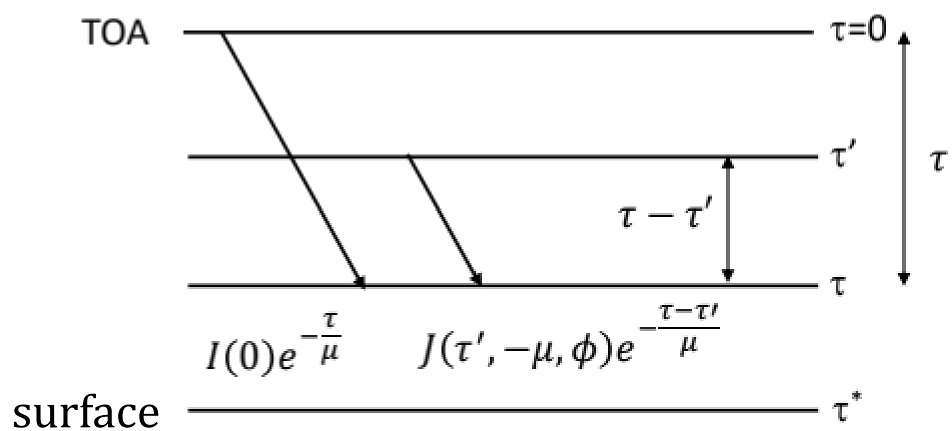


Solution for the Plane-Parallel Equation (7)

$$(\downarrow) \qquad I(\tau) = I(0)e^{-\tau/\mu} + \int_0^{\tau} J(\tau', -\mu, \phi)e^{-\frac{\tau-\tau'}{\mu}} d\left(\tau'/\mu\right)$$

The radiance that arrives at the level τ from above is given by

1. Arriving from the TOA reduced of the transmission due to the TOA-level τ layer
2. The contributions of the sources in all layers above level τ, each reduced by its relative transmission up to τ



Upward radiation at τ=0 (TOA)

If the radiance (↑) is observed at TOA $\tau = 0$ instead of at a generic level, we obtain the radiance that is observed outside the atmosphere, for example by an instrument mounted on a satellite for Earth observation (Earth Observation, EO)

$$(\uparrow) \qquad I(0) = I(\tau^*)e^{-\frac{\tau^*}{\mu}} + \int_0^{\tau^*} J(\tau', \mu, \phi)e^{-\frac{\tau'}{\mu}} d\left({\tau'}/{\mu}\right)$$

Contribution from surface reduced by the slant path transmission of whole atmosphere

Contribution from atmospheric internal layers, each reduced by slant transmission of the overlaying atmosphere

Useful for satellite remote sensing of surface!

Upward radiation at τ= τ* (surface)

If instead the radiance ↓ is observed from the surface τ=τ* instead of a generic level τ, we obtain the radiance measured by instruments housed on the ground which they look at the atmosphere or space.

(↓) $$I(\tau^*) = I(0)e^{-\tau^*/\mu} + \int_0^{\tau^*} J(\tau', -\mu, \phi)e^{-\frac{\tau^*-\tau'}{\mu}} d(\tau'/\mu)$$

Contribution from TOA reduced by the slant path transmission of whole atmosphere	Contribution from atmospheric internal layers, each reduced by slant transmission of the underlaying atmosphere

Useful for ground based remote sensing of space!